

Some facts about discriminants

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Abstract

For the family of polynomials in one variable $P := x^n + a_1x^{n-1} + \dots + a_n$ we ask the questions at which points its discriminant set can be considered as the graph of a function of all coefficients a_j but one and how its subset of points, where the discriminant set is not smooth, projects on the different coordinate hyperplanes in the space of the coefficients a .

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1 Introduction

Consider the family of polynomials $P := x^n + a_1x^{n-1} + \dots + a_n$ in the variable $x \in \mathbb{C}$. Set $a := (a_1, \dots, a_n)$ and $a^k := (a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n)$. Denote by $D := \{a \in \mathbb{C}^n \mid \text{Res}(P, P', x) = 0\}$ its *discriminant set*, i. e. the set of values of a for which P has a multiple root. In the present text we treat the question at which points the set D can be considered as the graph of a function in the variables a^j (and for which j) and how the set of its points, where it is not smooth, projects on the different coordinate hyperplanes in the space of the variables a . The results of this paper are applicable to the case $x, a_j \in \mathbb{R}$, when it is important to know the number of positive and negative real roots and the signs of the coefficients of P , see [1], [2] and the references therein. Some recent results about real discriminant sets can be found in [3].

The following result is known; we include its proof for the sake of completeness:

Lemma 1. *The set D is the zero set of an irreducible polynomial R in $a \in \mathbb{C}^n$. When considered as a polynomial in a_k , $k \leq n-1$, it is of degree n , with leading coefficient $\pm k^k(n-k)^{n-k}a_n^{n-k-1}$. The polynomial R is degree $n-1$ in a_n . One has $R|_{a_{n-1}=a_n=0} \equiv 0$.*

Remark 2. One can assign j as quasi-homogeneous weight to the coefficient a_j , $1 \leq j \leq n$ (because a_j is a symmetric degree j polynomial in the roots of P). It is well-known that R is a quasi-homogeneous polynomial in a of quasi-homogeneous degree $n(n-1)$. It equals $\pm \prod_{1 \leq i < j \leq n} (x_i - x_j)^2$, where x_i, x_j are the roots of P . In this product each difference is squared because when the coefficients a_j take values corresponding to a loop in the space \mathbb{C}^n circumventing the zero set of R , then generically two of the roots are interchanged.

Notation 3. (1) We denote by Σ the subset of D on which the corresponding polynomial P has a root of multiplicity at least 3 and by Σ_k its projection in the space \mathbb{C}_k^{n-1} of the variables a^k . Derivations w. r. t. x and a_k are denoted respectively by $'$ and $\partial/\partial a_k$.

(2) We denote by $S(F_1, F_2)$ the *Sylvester matrix* of the polynomials $F_1 = d_0x^{n_1} + d_1x^{n_1-1} + \dots + d_{n_1}$ and $F_2 = g_0x^{n_2} + g_1x^{n_2-1} + \dots + g_{n_2}$ (considered as polynomials in the variable x). The matrix $S(F_1, F_2)$ is $(n_1 + n_2) \times (n_1 + n_2)$ and its first (resp. its $(n_2 + 1)$ st) row equals

$$(d_0, d_1, \dots, d_{n_1}, 0, \dots, 0) \quad , \quad \text{resp.} \quad (g_0, g_1, \dots, g_{n_2}, 0, \dots, 0) \quad ,$$

its second and $(n_2 + 2)$ nd rows are obtained by shifting these ones by one position to the right while adding a zero in the first position etc. For polynomials G_1, G_2 in a_k , with coefficients in $\mathbb{C}[a^k]$, we write $S(G_1, G_2, a_k)$ for their Sylvester matrix.

(3) We denote by D_k the subset in the space \mathbb{C}_k^{n-1} of the variables a^k defined by the condition $\tilde{D}_k := \text{Res}(R, \partial R / \partial a_k, a_k) = 0$. When R is considered as a polynomial in the variable a_k with coefficients in $\mathbb{C}[a^k]$, the set D_k is the subset of \mathbb{C}_k^{n-1} on which R has a multiple root.

Proof of Lemma 1. Consider the matrix $S(P, P')$, see Notation 3. To simplify the computation of its determinant we subtract, for $j = 1, \dots, n-1$, its $(n-1+j)$ th row multiplied by $1/(n-k)$ from the j th one. We denote by S^* the newly obtained matrix. The only product of $2n-1$ entries in the determinant of S^* which contains n factors a_k is the following one (we list to the left the entries and to the right, for each entry, the positions in which it is encountered):

$$\begin{array}{ccccccc} -k/(n-k) & (1, 1) & (2, 2) & \dots & (k, k) & & \\ & a_n & (k+1, k+n+1) & (k+2, k+n+1) & \dots & (n-1, 2n-1) & \\ & (n-k)a_k & (n, k+1) & (n+1, k+2) & \dots & (2n-1, k+n) & \end{array}$$

Up to a sign the product equals $k^k(n-k)^{n-k}a_n^{n-k-1}a_k^n$.

The matrix $S(P, P')$ contains $n-1$ terms a_n , in positions $(j, j+n)$, $j = 1, \dots, n-1$. The product of these terms and of the constant terms in positions $(j+n-1, j)$, $j = 1, \dots, n$ give the only monomial Aa_n^{n-1} , $A \neq 0$, in $\det S(P, P')$. Irreducibility of R follows from the fact that a quasi-homogeneous polynomial with quasi-homogeneous weight j of the variable a_j (see Remark 2) and containing monomials Aa_n^{n-1} and Ba_{n-1}^n , $B \neq 0$, cannot be represented as a product of two nonconstant quasi-homogeneous polynomials. The equality $R|_{a_{n-1}=a_n=0} \equiv 0$ follows from $S(P, P')$ having as nonzero entries in its last column only a_n and a_{n-1} . \square

2 Where is the discriminant locally the graph of a function?

Theorem 4. (1) Suppose that at a point $A \in D$ the corresponding polynomial P has a double root λ and $n-2$ simple roots. Then

(i) if in addition $\lambda \neq 0$ (hence one does not have $a_{n-1} = a_n = 0$), then for $k = 1, \dots, n$, at this point the set D is locally the graph of an analytic function in a^k ;

(ii) if $\lambda = 0$, then this property holds true only for $k = n$ and fails for $k = 1, \dots, n-1$.

(2) Suppose that at a point $A \in D$ the corresponding polynomial P has a root of multiplicity at least 3. Then at this point the hypersurface D is not smooth.

Proof of Theorem 4. Set $P = (x + \lambda)^2 Q$, $Q := (x^{n-2} + b_1 x^{n-3} + \dots + b_{n-2})$. Hence

$$\begin{aligned} a_1 &= 2\lambda + b_1 \quad , & a_2 &= \lambda^2 + 2\lambda b_1 + b_2 \quad , & a_3 &= \lambda^2 b_1 + 2\lambda b_2 + b_3 \quad , \quad \dots \quad , \\ a_{n-2} &= \lambda^2 b_{n-4} + 2\lambda b_{n-3} + b_{n-2} \quad , & a_{n-1} &= \lambda^2 b_{n-3} + 2\lambda b_{n-2} \quad , & a_n &= \lambda^2 b_{n-2} \quad . \end{aligned}$$

Consider the $(n-1) \times n$ matrix $\tilde{J} := (\partial(a_1, a_2, \dots, a_n) / \partial(\lambda, b_1, \dots, b_{n-2}))^T$. It equals

$$\begin{pmatrix} 2 & 2\lambda + 2b_1 & 2\lambda b_1 + 2b_2 & 2\lambda b_2 + 2b_3 & \cdots & 2\lambda b_{n-4} + 2b_{n-3} & 2\lambda b_{n-3} + 2b_{n-2} & 2\lambda b_{n-2} \\ 1 & 2\lambda & \lambda^2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2\lambda & \lambda^2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2\lambda & \lambda^2 \end{pmatrix}$$

We denote by C_k the k th column of the matrix \tilde{J} and by J_k its submatrix obtained by deleting C_k . As $Q(-\lambda) \neq 0$, all claims of the theorem follow from the following lemma:

Lemma 5. *One has $\det J_k = (-1)^n 2\lambda^{n-k} Q(-\lambda)$.*

□

Proof of Lemma 5. Denote by S_k the $(n-1)$ -vector-column $(2b_{k-1}, 0, \dots, 0, \lambda, 1, 0, \dots, 0)^T$, where $2 \leq k \leq n-1$ and λ is preceded by $k-2$ zeros. We set $S_1 := (2, 1, 0, \dots, 0)^T$, $S_0 := (0, \dots, 0)^T$. Hence $C_k = \lambda S_{k-1} + S_k$. Set $J_k := J'_k + J''_k$, where

$$J'_k = \lambda(C_1, \dots, C_{k-1}, S_k, C_{k+2}, \dots, C_n) \quad , \quad J''_k = (C_1, \dots, C_{k-1}, S_{k+1}, C_{k+2}, \dots, C_n) \quad .$$

Thus $\det J_k = \lambda \det J'_k + \det J''_k$. One has $\det J''_k = 0$. Indeed,

$$\det J''_k = \det(C_1, \dots, C_{k-1}, S_{k+1}, \lambda S_{k+1} + S_{k+2}, \lambda S_{k+2} + S_{k+3}, \dots, \lambda S_{n-2} + S_{n-1}, \lambda S_{n-1}) \quad .$$

Subtract consecutively for $j = k, \dots, n-2$, the j th column multiplied by λ from the $(j+1)$ st one. This does not change the determinant. After these subtractions all entries of the last column are zeros, so $\det J''_k = 0$ and $\det J_k = \lambda \det J'_k$. After this set $J'_k := J^*_k + J^{**}_k$, where

$$J^*_k = \lambda^2(C_1, \dots, C_{k-1}, S_k, S_{k+1}, C_{k+3}, \dots, C_n) \quad , \quad J^{**}_k = \lambda(C_1, \dots, C_{k-1}, S_k, S_{k+2}, C_{k+3}, \dots, C_n) \quad .$$

By analogy with $\det J''_k = 0$ we show that $\det J^{**}_k = 0$. Hence $\det J_k = \lambda^2 \det J^*_k$. Continuing like this we conclude that

$$\begin{aligned} \det J_k &= \lambda^{n-k} \det(C_1, \dots, C_{k-1}, S_k, S_{k+1}, \dots, S_{n-1}) \\ &= \lambda^{n-k} \det(S_1, \lambda S_1 + S_2, \dots, \lambda S_{k-2} + S_{k-1}, S_k, S_{k+1}, \dots, S_{n-1}) \quad . \end{aligned}$$

We subtract then consecutively (for $j = 1, \dots, k-2$) the j th column multiplied by λ from the $(j+1)$ st one. This does not change $\det J_k$, so

$$\det J_k = \lambda^{n-k} \Delta \quad , \quad \Delta := \det(S_1, S_2, \dots, S_{k-1}, S_k, S_{k+1}, \dots, S_{n-1}) \quad . \quad \text{i. e.}$$

$$\Delta = \begin{vmatrix} 2 & 2b_1 & 2b_2 & 2b_3 & \cdots & 2b_{n-3} & 2b_{n-2} \\ 1 & \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \lambda \end{vmatrix}$$

To compute Δ we subtract consecutively (for $j = 1, \dots, n-2$) the j th column multiplied by λ from the $(j+1)$ st one. We get

$$\Delta = 2 \begin{vmatrix} 1 & b_1 - \lambda & b_2 - b_1\lambda + \lambda^2 & \cdots & b_{n-3} - \lambda b_{n-4} + \cdots + (-1)^{n-3}\lambda^{n-3} & Q(-\lambda) \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}$$

Hence $\Delta = (-1)^n 2Q(-\lambda)$. \square

Remarks 6. (1) Consider the product $P_1 P_2$, where $P_1 := x^m + b_1 x^{m-1} + \cdots + b_m$, $P_2 := x^l + c_1 x^{l-1} + \cdots + c_l$ and b_i, c_j are complex parameters. Suppose that for some value of these parameters the polynomials P_1 and P_2 have no root in common. Then locally (close to this value) the discriminant set of $P_1 P_2$ (defined by analogy with D) is diffeomorphic to the direct product of the discriminant sets of P_1 and P_2 at the respective values of b_i and c_j . This follows from the Lemma about the product on p. 12 of [4]. (In [4] the author considers the case of real polynomials, but the proof of the Lemma about the product is carried out in the complex case in exactly the same way as in the real one.)

(2) Suppose that at a point $A \in D$ the polynomial P has k double roots and $n - 2k$ simple ones. Then locally, at A , the set D is the transversal intersection of k analytic hypersurfaces each of which satisfies statements (i) and (ii) of Proposition 4. Transversality follows from part (1) of the present remarks, the analogs of statements (i) and (ii) of the proposition are proved in exactly the same way as the proposition itself (in the proof of the proposition we do not use the fact that the rest of the roots of P are simple).

Proposition 7. (1) For $k \leq n-1$ the polynomial \tilde{D}_k is not divisible by a_i for $k \neq i \neq n$.

(2) The polynomial \tilde{D}_n is not divisible by a_i for $i \leq n-1$.

(3) If $k \leq n-2$, then \tilde{D}_k is divisible by a_n^{n-k-1} and not divisible by a_n^{n-k} .

(4) The polynomial \tilde{D}_{n-1} is divisible by a_n and not divisible by a_n^2 .

Proof of Proposition 7. Throughout the proof the letter Ω (indexed or not) stands for nonspecified nonzero constants. We set $T_k := S(R, \partial R / \partial a_k, a_k)$. Recall that T_k is $(2n-1) \times (2n-1)$ for $k \leq n-1$ and $(2n-3) \times (2n-3)$ for $k = n$, see Lemma 1.

Statement 8. *For $a_i = 0$, $k \neq i \neq n$, $k < n$, one has $R = \Omega_1 a_k^n a_n^{n-k-1} + \Omega_2 a_n^{n-1}$.*

Proof. Indeed, in this case one computes R easily if one subtracts for $j = 1, \dots, n-1$ the $(n-1+j)$ th row of $S(P, P')$ multiplied by $1/(n-k)$ from its j th one. This doesn't change $\det S(P, P') = R$ and the matrix obtained from $S(P, P')$ has only the following nonzero entries:

$$\begin{array}{ccc} -k/(n-k) & (j, j) & , \quad a_n & (j, j+n) \\ \text{in positions} & & & \text{in positions} \\ n & (n-1+\nu, \nu) & , \quad (n-k)a_k & (n-1+\nu, \nu+k) \end{array}$$

$(1 \leq j \leq n-1, 1 \leq \nu \leq n)$. To prove Statement 8 with this form of $\det S(P, P')$ is easy. \square

Statement 8 implies that for $a_i = 0$, $k \neq i \neq n$, $k < n$, the matrix T_k has nonzero entries only $\Omega_1 a_n^{n-k-1}$, $\Omega_2 a_n^{n-1}$ and $n\Omega_1 a_n^{n-k-1}$, respectively in positions (j, j) , $(j, j+n)$, $j = 1, \dots, n-1$, and $(n-1+\nu, \nu)$, $\nu = 1, \dots, n$. Clearly $\det T_k = \Omega_3 a_n^{(n-1)^2 + n(n-k-1)} \neq 0$. Hence \tilde{D}_k is not divisible by a_i for $i \neq n$. Part (1) is proved.

To prove part (2) for $i < n-1$ we use Statement 8 with $k = n-1$. Hence the $(2n-3) \times (2n-3)$ -matrix $T_{n-1}|_{a_i=0, n-1 \neq i \neq n}$ has nonzero entries only

$$\begin{array}{ccc} \Omega_2 & (j, j) & , \quad \Omega_1 a_{n-1}^n \text{ in positions } (j, n-1+j) \\ \text{in positions} & & \\ (n-1)\Omega_2 & (n-2+\nu, \nu) & \end{array}$$

$(1 \leq j \leq n-2, 1 \leq \nu \leq n-1)$. It is easy to show that its determinant equals $\Omega_4 a_{n-1}^{n(n-2)} \neq 0$.

To prove part (2) for $i = n-1$ we apply Statement 8 with $k = n-2$. Hence T_n has nonzero entries only (with j and ν as above)

$$\begin{array}{ccc} \Omega_2 & (j, j) & , \quad \Omega_1 a_{n-2}^n & (j, n-2+j) \\ \text{in positions} & & & \text{in positions} \\ (n-1)\Omega_2 & (n-2+\nu, \nu) & , \quad \Omega_1 a_{n-2}^n & (n-2+\nu, n-2+\nu) \end{array}$$

Its determinant equals $\Omega_5 a_{n-2}^{n(n-1)} \neq 0$. Part (2) is proved.

To prove part (3) consider the $(2n-1) \times (2n-1)$ -matrix T_k . It has two entries in its first column, in positions $(1, 1)$ and $(n, 1)$. By Lemma 1 they are of the form Ωa_n^{n-k-1} and $n\Omega a_n^{n-k-1}$. This proves the first statement of part (3). To prove its second statement consider for $k \leq n-2$ the polynomial $P^0 := P|_{a_i=0, k \neq i \neq n-1} = x^n + a_k x^{n-k} + a_{n-1} x$. Set $\delta := \det S(P^0, P^{0'})$, $\delta_1 := \det S(P^0/x, P^{0'})$.

Statement 9. *One has $\delta = a_{n-1} \delta_1 = \Omega_5 a_{n-1}^n + \Omega_6 a_k^{n-1} a_{n-1}^{n-k}$.*

Proof. In its last column the matrix $S(P^0, P^{0'})$ has a single nonzero entry (namely a_{n-1} , in position $(2n-1, 2n-1)$), so $\delta = a_{n-1} \delta_1$. To compute easily δ_1 we subtract for $j = n, n+1, \dots, 2n-2$ the j th row of $S(P^0/x, P^{0'})$ from its $(j-n+1)$ st row. This makes disappear the terms a_{n-1} in the first $n-1$ rows. After the subtractions the first $n-1$ rows have entries Ω_* in positions (j, j) , $\Omega'_k a_k$ in positions $(j, j+k)$ and zeros elsewhere.

Then we subtract for $i = 1, \dots, n-1$ the i th row multiplied by a constant from the $(i+n-1)$ st one. The constant is chosen such that after the subtraction the $(i+n-1)$ st row contains no term $\Omega^* a_k$. After all these subtractions the matrix obtained from $S(P^0/x, P^{0'})$ has nonzero terms in the following positions (and zeros elsewhere):

$$\begin{array}{ccc} \Omega_* & (j, j) & , \quad \Omega' a_k & (j, j+k) \\ \text{in positions} & & & \text{in positions} \\ \Omega_{**} & (j+n-1, j) & , \quad \Omega_{***} a_{n-1} & (j+n-1, j+n-1) \end{array}$$

$(1 \leq j \leq n-1)$. Hence $\delta = a_{n-1} \delta_1 = \Omega_5 a_{n-1}^{n-1} + \Omega_6 a_k^{n-1} a_{n-1}^{n-k}$. \square

Thus the matrix $T_k|_{a_i=0, k \neq i \neq n-1}$ has only the following nonzero entries:

$$\begin{array}{ccc} \Omega_6 a_{n-1}^{n-k} & (j, j+1) & , \quad \Omega_5 a_{n-1}^n & \text{in positions} & (j, n+j) \\ \text{in positions} & & & & \\ (n-1) \Omega_6 a_{n-1}^{n-k} & (n-1+\nu, \nu+1) & , & 1 \leq j \leq n-1 & , & 1 \leq \nu \leq n . \end{array}$$

We consider the expansion of $\det T_k$ in a series in a_n . Our aim is to show that the initial term equals $W a_n^{n-k-1}$ with $W \neq 0$ with which part (3) will be proved. When expanding $\det T_k$ w. r. t. its first column (where it has just two nonzero entries) one gets

$$\det T_k = (-1)^{1+1} \Omega a_n^{n-k-1} \det(T_k)_{1,1} + (-1)^{n+1} n \Omega a_n^{n-k-1} \det(T_k)_{n,1} ,$$

where $(T_k)_{p,q}$ means the matrix obtained from T_k by deleting its p th row and its q th column. In the first column of the matrix $(T_k)_{1,1}$ (resp. $(T_k)_{n,1}$) only the entry in position $(n-1, 1)$ (resp. $(1, 1)$) does not vanish for $a_n = 0$, see Lemma 1 and Statement 9. For $a_i = 0$, $k \neq i \neq n-1$, this entry equals $(n-1) \Omega_6 a_{n-1}^{n-k}$ (resp. $\Omega_6 a_{n-1}^{n-k}$, see Statement 9). Thus

$$\det(T_k)_{1,1} = (-1)^{n-1+1} (n-1) \Omega_6 a_{n-1}^{n-k} (\det((T_k)_{1,1})_{n-1,1}|_{a_n=0}) + O(\sum_{k \neq i \neq n-1} |a_i|) ,$$

$$\det(T_k)_{n,1} = (-1)^{1+1} \Omega_6 a_{n-1}^{n-k} (\det((T_k)_{n,1})_{1,1}|_{a_n=0}) + O(\sum_{k \neq i \neq n-1} |a_i|) .$$

Now observe that each of the matrices

$$((T_k)_{1,1})_{n-1,1}|_{a_i=0, k \neq i \neq n-1} \quad \text{and} \quad ((T_k)_{n,1})_{1,1}|_{a_i=0, k \neq i \neq n-1}$$

is equal to the Sylvester matrix $S(\Omega_5 a_{n-1}^n + \Omega_6 a_k^{n-1} a_{n-1}^{n-k}, \Omega_6 (n-1) a_k^{n-2} a_{n-1}^{n-k}, a_k) =: S'$ whose determinant is of the form $\Omega_{***} a_{n-1}^{n(n-2)+(n-k)(n-1)} \neq 0$, therefore

$$\begin{aligned} W_{a_i=0, k \neq i \neq n-1} &= \Omega \Omega_6 a_{n-1}^{n-k} ((-1)^{1+1} (-1)^{n-1+1} (n-1) + (-1)^{n+1} (-1)^{1+1} n) \det S' \\ &= \Omega \Omega_6 a_{n-1}^{n-k} (-1)^{n-1} \det S' \neq 0 . \end{aligned}$$

This proves part (3). To prove part (4) we need

Statement 10. *The polynomial R is of the form $a_{n-1}^2 U(a) + a_n V(a)$, where U and $V|_{a_n=0} \neq 0$ are polynomials.*

Proof. Set $S^0 := S(P, P')|_{a_n=0}$. The only nonzero entry in the last column of S^0 is a_{n-1} in position $(2n-1, 2n-1)$, so $R|_{a_n=0} = a_{n-1} \det((S^0)_{2n-1, 2n-1})$. Both nonzero entries in the last column of $(S^0)_{2n-1, 2n-1}$ equal a_{n-1} , so $R|_{a_n=0}$ is divisible by a_{n-1}^2 hence $R = a_{n-1}^2 U(a) + a_n V(a)$. There remains to show that $V|_{a_n=0} \neq 0$. To this end consider the matrix $M := S(P, P')|_{a_{n-1}=0}$. One has $\det M = (-1)^{n-1+2n-1} a_n \det((M)_{n-1, 2n-1})$. For $a_i = 0$, $i \neq n-2$, the only nonzero entries of $(M)_{n-1, 2n-1}$ are

$$\begin{array}{ccc} 1 & (j, j) & , \quad a_{n-2} & (j, n-2+j) \\ \text{in positions} & & & \text{in positions} \\ n & (n-2+\nu, \nu) & , \quad 2a_{n-2} & (n-2+\nu, n-2+\nu) \end{array}$$

$(1 \leq j \leq n-2, 1 \leq \nu \leq n-1)$. Hence $\det((M)_{n-1, 2n-1})|_{a_i=0, i \neq n-2} = \Omega^\omega a_{n-2}^{n-1} \neq 0$. \square

Statement 10 implies that the last column of $T_{n-1}|_{a_n=0}$ contains only zeros, so \tilde{D}_{n-1} is divisible by a_n . Denote by $t_{i,j}$ the entry of T_{n-1} in position (i, j) . Hence

$$\det T_{n-1} = (-1)^{3n-5} t_{n-2, 2n-3} \det((T_{n-1})_{n-2, 2n-3}) + (-1)^{4n-6} t_{2n-3, 2n-3} \det((T_{n-1})_{2n-3, 2n-3})$$

The terms $t_{n-2, 2n-3}$ and $t_{2n-3, 2n-3}$ are divisible by a_n and $t_{n-2, 2n-3} = a_n V(a)$ is not divisible by a_n^2 . All entries in the last column of $(T_{n-1})_{2n-3, 2n-3}$ are divisible by a_n (see Statement 10), so $t_{2n-3, 2n-3} \det((T_{n-1})_{2n-3, 2n-3}) = O(a_n^2)$. On the other hand the nonzero entries of the matrix $\tilde{T} := (T_{n-1})_{n-2, 2n-3}|_{a_i=0, i \neq n-2, n-1}$ are (see Statement 9 with $k = n-2$)

$$\begin{array}{ccc} \Omega_5 & (j, j) & , \quad \Omega_6 a_{n-2}^{n-1} & (j, n-2+j) \\ \text{in positions} & & & \text{in positions} \\ n\Omega_5 & (n-2+\nu, \nu) & , \quad 2\Omega_6 a_{n-2}^{n-1} & (n-2+\nu, n-2+\nu) \end{array}$$

$(1 \leq j \leq n-2, 1 \leq \nu \leq n-1)$. Hence $\det \tilde{T} = \Omega_\lambda a_{n-2}^{(n-1)^2} \neq 0$. Thus $t_{n-2, 2n-3} \det((T_{n-1})_{n-2, 2n-3})$ (and hence $\det T_{n-1}$ as well) is divisible by a_n , but not by a_n^2 . This proves part (4). \square

3 The projections of the set Σ

The set Σ (see Notation 3) is a codimension 2 algebraic subset of \mathbb{C}^n . It is locally defined by the equations of any two of its projections Σ_k , or by $\text{Res}(P, P', x) = \text{Res}(P', P'', x) = 0$. Consider for $1 \leq k \leq n-1$ the polynomial $P_k := P - xP'/(n-k)$; its coefficient of x^{n-k} equals 0. When P has a root $\alpha \neq 0$ of multiplicity $m \geq 1$, then α is a root of P_k of multiplicity $m-1$. We set $P_n := P'$.

Theorem 11. *For $k \neq n-1$ the polynomial $V_k := \text{Res}(P_k, P'_k, x)$ is irreducible. The polynomial $\text{Res}(P_{n-1}, P'_{n-1}, x)$ is the product of a_n and of an irreducible polynomial in a^{n-1} ; we set $V_{n-1} := \text{Res}(P_{n-1}, P'_{n-1}, x)/a_n$.*

Theorem 12. (1) *For $k = 1, \dots, n$ the set Σ_k is defined by the condition $V_k = 0$.*
 (2) *The sets Σ_k are irreducible.*

Proof of Theorem 11. Irreducibility of V_n follows from an analog of Lemma 1 formulated for P' instead of P . For $1 \leq k \leq n-2$ the polynomial V_k is a quasi-homogeneous polynomial in a^k containing monomials $A_k a_n^{n-1}$ and $B_k a_{n-1}^n$ (resp. $A_n a_{n-1}^{n-2}$ and $B_n a_{n-2}^{n-1}$), where $A_k \neq 0$ and

$B_k \neq 0$ are constants, see Lemma 1 and its proof. Such a polynomial cannot be represented as a product of two quasi-homogeneous polynomials of smaller quasi-homogeneous degrees because the quasi-homogeneous weights of a_n, a_{n-1} (resp. of a_{n-1} and a_{n-2}) equal n and $n-1$ (resp. $n-1$ and $n-2$). (If such a representation exists, and if h_1 and h_2 are the quasi-homogeneous degrees of the two factors, then the monomial $A_k a_n^{n-1}$ is a product of monomials $H^1 a_n^{k_1}$ and $H^2 a_n^{k_2}$ of each factor and $k_1 n = h_1, k_2 n = h_2, k_1 < n-1, k_2 < n-1$. In the same way the monomial $B_k a_{n-1}^n$ implies the existence of $l_1, l_2 \in \mathbb{N}$ such that $l_1(n-1) = h_1 = k_1 n, l_2(n-1) = h_2 = k_2 n, l_1 < n, l_2 < n$ which is impossible.)

The polynomial $R_{n-1} := \text{Res}(P_{n-1}, P'_{n-1}, x)$ is reducible. This follows from

Property A. *The nonzero entries in the last two columns of the matrix $S := S(P_{n-1}, P'_{n-1})$ are $s_{n-2, 2n-2} = s_{n-1, 2n-1} = a_n$ and $s_{2n-1, 2n-2} = -2a_{n-2}$.*

Hence every monomial of R_{n-1} is divisible by a_n , and R_{n-1}/a_n (of quasi-homogeneous degree $n(n-2)$) contains monomials of the form $C_j a_j^n a_n^{n-j-2}$, $C_j \neq 0, j = 1, 2, \dots, n-2, n$ (this is proved by complete analogy with Lemma 1). If n is odd, then $(n, n-2) = 1$ and applying to the monomials $C_{n-2} a_{n-2}^n$ and $C_n a_n^{n-2}$ a reasoning similar to the one above (which was applied to $A_k a_n^{n-1}$ and $B_k a_{n-1}^n$) one concludes that R_{n-1}/a_n is irreducible.

If n is even, then R_{n-1}/a_n could be reducible only if it is the product of two polynomials of quasi-homogeneous degree $n(n-2)/2$. To show that this is impossible consider the monomial $C_{n-3} a_{n-3}^n a_n$ (for which $C_{n-3} \neq 0$, see Lemma 1). It must be the product of monomials $C'_{n-3} a_{n-3}^s$ and $C''_{n-3} a_{n-3}^q a_n$. Hence $s(n-3) = n(n-2)/2 = q(n-3) + n$, i.e. $s = n(n-2)/2(n-3)$ and $q = n(n-4)/2(n-3)$. The numbers $n-2$ and $n-3$ are coprime, and such are $n-4$ and $n-3$ as well. For $n > 6$ the ratio $n/(n-3)$ is not integer, so such a product of polynomials (equal to R_{n-1}/a_n) does not exist. In the particular cases $n = 4$ and $n = 6$ one can check with the help of a computer that the corresponding polynomials P_3/a_4 and P_5/a_6 are irreducible. \square

Proof of Theorem 12. For $k \neq n-1$ the polynomial V_k defines the subset $Z_k \subset \mathbb{C}_k^{n-1}$ on which the polynomial P_k has a multiple root. Hence $\Sigma_k \subset Z_k$. On the other hand the set Z_k is irreducible (see Theorem 11) and of codimension 1 in the space \mathbb{C}_k^{n-1} , therefore $\Sigma_k = Z_k$. For $k \neq n-1$ part (2) follows from the irreducibility of the polynomials V_k , see Theorem 11.

Let $k = n-1$. The polynomial V_{n-1} is of the form $a_{n-2}W^* + a_n W^{**}$, $W^*, W^{**} \in \mathbb{C}[a^{n-1}]$, see Property A. If 0 is a triple root of P , then $a_{n-2} = a_{n-1} = a_n = 0$ and the projection of this point of Σ in the space \mathbb{C}_{n-1}^{n-1} belongs to the set $\{V_{n-1} = 0\}$. If $\beta \neq 0$ is a triple root of P , then $P = (x - \beta)^3(x^{n-3} + g_1 x^{n-4} + \dots + g_{n-3})$, where the coefficients g_j run over a whole neighbourhood in \mathbb{C}^{n-3} . Such points of Σ project in \mathbb{C}_{n-1}^{n-1} in the set $\{V_{n-1} = 0\}$, but not in $\{a_n = 0\} \setminus \{V_{n-1} = 0\}$. This proves the theorem for $k = n-1$. \square

References

- [1] A. Albouy and Y. Fu, Some Remarks About Descartes Rule of Signs, *Elemente der Mathematik* 69 (2014), 186194.
- [2] J. Forsgård, V.P. Kostov and B.Z. Shapiro, Could René Descartes have known this?, *Experimental Mathematics* vol. 24, issue 4 (2015) 438-448.
- [3] V.P. Kostov, Topics on hyperbolic polynomials in one variable. *Panoramas et Synthèses* 33 (2011), vi + 141 p. SMF.
- [4] I. Méguerditchian, *Géométrie du Discriminant Réel et des Polynômes Hyperboliques*, Thèse de Doctorat (soutenue le 24 janvier 1991 à Rennes).